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Discussion

Comments on the historical bases of the Rayleigh and Ritz methods

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In a recent article [1] entitled "The historical bases of the Rayleigh and Ritz methods", Professor Leissa raised the issue of using two different names, "The Rayleigh-Ritz Method" and "The Ritz Method" for the same procedure. While the discusser shares Professor Leissa's view that the success of the method can be attributed to the way in which it was applied by Ritz, he does not endorse the position that it should not be referred to as the Rayleigh-Ritz method for the reasons given below.

Lord Rayleigh's writings [2,3] contain many logical statements written without using mathematical expressions or equations. There are statements which require one to infer the reasoning. For example, Lord Rayleigh does not say how he obtained the minimum natural frequency of a vibrating string with an assumed symmetrical deflected form $1-(2x/l)^n$ defined over its half-length from the centreline, but gives the correct expression and states clearly that this is a minimum. As Professor Leissa has stated in Ref. [1], these omissions may have been either because the missing steps and mathematical statements were too obvious for Lord Rayleigh, or in his view, too obvious to the intended readership of that time. From the examples presented in Refs. [3–5], it is clear that Lord Rayleigh had also used a series approximation to obtain an upperbound estimate of the fundamental frequency by minimisation, although the number of adjustable terms (i.e. degrees of freedom) in the series was limited to two and the type of minimisation was not as convenient as the one subsequently proposed by Ritz [6,7].

Professor Leissa makes the following statement in page 963 of [1]: "Perhaps because Rayleigh himself claimed to deserve sharing credit for the Ritz method, many subsequent researchers attached his name to it, without looking for verification." The discusser has a different view, at least when considering the use of the name the "Rayleigh-Ritz Method" by the late Professor Richard Courant.

Courant (1888–1972) was a brilliant mathematician of his time and a contemporary of David Hilbert (1862–1943) and Hillbert's one time student Walther Ritz (1878–1909). His address to the American Mathematical Society in 1943 [8] contains several significant propositions that have helped to shape computational mechanics in the following decades. These include: the concept of artificial stiffness to effect geometric constraints which has subsequently led to what is now known as the penalty method, and the piecewise representation of admissible functions in finite elements. In this address, published in the Bulletin of

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the American Mathematical Society, Courant devoted a section to the Rayleigh-Ritz Method [8]. He gives credit to both Rayleigh and Ritz for the method. The following paragraph is extracted from this paper.

> Since Gauss and W. Thompson, the equivalence between boundary value problems of partial differential equations on the one hand and problems of the calculus of variations on the other hand has been a central point in the analysis. At first, the theoretical interests in existence proofs dominated and only much later were practical applications envisaged by two physicists, Lord Rayleigh and Walther Ritz; they independently conceived the idea of utilizing this equivalence for numerical calculation of the solutions, by substituting for the variational problems simpler approximating extremum problems in which but a finite number of parameters need to be determined. Rayleigh, in his classical work—*Theory of Sound*—and in other publications, was the first to use such a procedure. But only the spectacular success of Walther Ritz and its tragic circumstances caught the general interest. In two publications of 1908 and 1909 [6,7], Ritz, conscious of his imminent death from consumption, gave a masterly account of the theory, and at the same time applied his method to the calculation of the nodal lines of vibrating plates, a problem of classical physics that previously had not been satisfactorily treated.

When describing this method under the heading the ''Rayleigh-Ritz Method'', Courant refers to the specific procedure discovered by Ritz as follows: ''… the problem in applications is one, not of existence, but of practical construction of such a minimising sequence. Ritz's method is nothing but a recipe for such a construction of a minimising sequence''.

Courant has been consistent in giving credit to Rayleigh for the minimisation procedure. In the first volume of the well-known book ''Methods of Mathematical Physics'' by Courant and Hilbert [9], the statement that ''The method which W. Ritz[1] used with great success…'' is linked to a footnote which states ''Even before Ritz, similar ideas were successfully employed by Lord Rayleigh.''

It was Lord Rayleigh who first showed how to find good estimates of the fundamental natural frequency by minimising the frequency expression with respect to certain undetermined parameters. The Rayleigh-Ritz Method as it is commonly used now, and as used by Courant and his contemporaries, is an enhanced version of this minimisation procedure. It was Walther Ritz who showed how to do the minimisation more elegantly, paving the way for the more convenient and general procedure which we now use frequently. This has been clearly acknowledged by Courant [8,9], Hilbert [9] and Rayleigh [10]. Therefore, it is quite appropriate to give credit to both these pioneers by referring to this method as the Rayleigh-Ritz Method.

While it must be acknowledged that most of the practical minimisation exercises are now done in the way shown by Ritz, the more general minimisation concept should not be forgotten. The possibility that, for certain problems, a non-Ritz-type minimisation could be more practical should not be ruled out. It should also be observed that a change of name implemented for the purpose of being very specific can also break the threads in literature surveys in the future.

The discusser has high regard for Professor Leissa, and felt honoured when he received an invitation from Professor Leissa to collaborate on this topic. Regretfully, the difference of opinion on the key issue of the name of the method made it impossible for the discusser to join as a co-author. However, the interactions with Professor Leissa have helped him to gain some interesting historical knowledge and a better understanding of the methods discussed, for which he wishes to record his appreciation here. The discusser also wishes to thank Professor Hagedorn (TUD, Germany) for translating parts of one of the papers by Walther Ritz.

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